Aeroelastic Stability of Slender, Spinning Missiles

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The complete coupled equations of motion that govern the static and dynamic aeroelastic stability of spinning, flexible missiles are derived. A Lagrangian approach is employed that yields terms that produce nonlinear coupling between the elastic deflections and the rigid-body motion parameters. The equations are reduced to a linear set for stability analysis. First-order aerodynamics are formulated for application to a hypersonic missile. Some numerical examples are given that reveal a destabilizing influence of structural damping at certain roll rates for a particular missile configuration.

Nomenclature

C_{m_q}	= aerodynamic pitch damping derivative
$C^*_{m_q}$	= aerodynamic pitch damping coefficient (positive for stable damping)
C_{N_α}	= aerodynamic normal force derivative
$C^*_{N_\alpha}$	= aerodynamic normal force coefficient
c	= damping coefficient
D	= damping energy
d	= aerodynamic reference diameter
E	= Young's modulus
EI	= flexural rigidity
F_T	= complex body-fixed trim force, $F_y^T + iF_z^T$
F_y, F_z	= resultant lateral forces in y, z directions, respectively
F_y^T, F_z^T	= resultant body-fixed trim forces in y, z directions, respectively
$f_y(x), f_z(x)$	= distributed lateral forces in y, z directions, respectively
$f_y^T(x), f_z^T(x)$	= distributed body-fixed trim force in y, z directions, respectively
I	= cross-sectional area moment of inertia; missile lateral mass moment of inertia
I_x	= missile roll mass moment of inertia
i	= $\sqrt{-1}$
$L_\alpha(x)$	= lift force derivative per unit length
M_i	= ith mode generalized mass
M_T	= complex body-fixed trim moment, $M_y^T + iM_z^T$
M_x, M_y, M_z	= applied moments about x,y,z, respectively
M_α	= aerodynamic pitch moment derivative
m	= missile mass
$m(x)$	= mass per unit length
p,q,r	= missile angular rates about the x,y,z axes, respectively (components of $\bar{\omega}$)
Q_i	= ith mode generalized force
q_i	= generalized coordinates
q_∞	= dynamic pressure
$\bar{r}$	= position vector from c.g. to point on deflected missile
$[\dot{\bar{r}}]_{\rm rel}$	= velocity of point on missile relative to rotating x,y,z axes
S	= aerodynamic reference area
s	= Laplace transform variable, $\sigma + i\omega$
T	= kinetic energy
t	= time
U	= elastic strain energy
u,v,w	= velocity components along the x,y,z axes, respectively
$\bar{v}$	= absolute velocity of point on missile
$\bar{v}_{cg}$	= velocity of moving origin
x,y,z	= body-fixed missile axes
x, δ_y, δ_z	= components of $\bar{r}$
α	= angle of attack, w/u
β	= angle of sideslip, v/u
δ	= complex elastic deflection, $\delta_y + i\delta_z$
δ_y, δ_z	= lateral elastic deflections
η_i, ζ_i	= generalized coordinates (deflection magnitudes)
μ_i	= ith mode structural damping ratio
ξ	= complex angle of attack, $\beta + i\alpha$
σ	= real part of s
$\phi_i(x)$	= ith normal mode shape
Ω	= complex lateral rate, $q + ir$
ω	= missile angular velocity; imaginary part of s
ω_i	= ith mode bending frequency

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I. Introduction

A RIGOROUS formulation of the equations of motion for a spinning, aeroelastic missile is based on the Lagrangian approach of Meirovitch and Nelson,[1] who examined the aeroelastic stability of a spin-stabilized spacecraft containing flexible rod appendages. Deflections of the flexible missile are treated as the motion of particles relative to a rotating coordinate frame. The resulting energy expressions contain nonlinear terms that represent coupling between the elastic deflections and the rigid-body motions. The equations are reduced to a linear set for stability analysis.

The aeroelastic stability of spinning, flexible missiles has been examined previously. Oberholtzer et al.[2] describe a linear equation set derived earlier[3] in which the rigid-body equations are formulated independently of the elastic bending equations. Crimi[4] also derives from Lagrange's equations the linear equations of motion for a spinning, aeroelastic missile. Structural damping is not included in Crimi's paper and is found in this study to have a strong influence on dynamic aeroelastic stability for certain roll rates. Meirovitch and Wesley[5] examined the dynamic characteristics of a nonspinning aeroelastic rocket with variable mass.

First-order aerodynamics are formulated for application to a hypersonic missile. The numerical evaluation requires bending modes and frequencies and axial variation of the lift-force derivative, in addition to the rigid-body aerodynamic coefficients, stability derivatives, and mass properties. Numerical examples are included that illustrate the destabilizing influence of viscous structural damping.

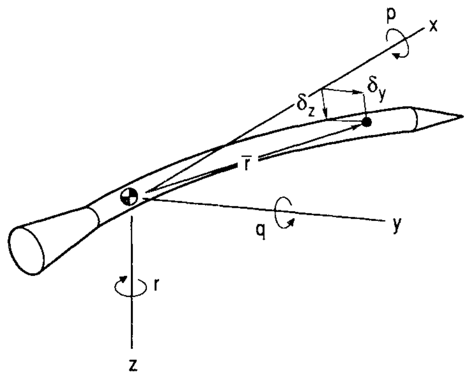

Fig. 1 Rotating coordinate frame.

II. Formulation

A. Bending Equations

Deflections of the flexible missile are treated as the motion of particles relative to a rotating coordinate frame (Fig. 1). The missile rotates in inertial space with angular velocity $\bar{\omega}$ having components p,q,r about the body-fixed axes x,y,z. We consider only the lateral deflections δ_y and δ_z in the y and z directions. The missile is assumed to be inextensible in the x direction. The position vector $\bar{r}$ from the center of gravity to a point on the deflected body has components x, δ_y, δ_z. The absolute velocity of the point under consideration, from kinematics, is given by (see, for example, Thomson[6])

$$\bar{v} = \bar{v}_{cg} + [\dot{\bar{r}}]_{\rm rel} + \bar{\omega} \times \bar{r} \tag{1}$$

where $\bar{v}_{cg}$ is the velocity of the moving origin; $[\dot{\bar{r}}]_{\rm rel}$ with components 0, $\dot{\delta}_y$, $\dot{\delta}_z$ is the velocity of the point relative to the rotating x,y,z axes; and $\bar{\omega} \times \bar{r}$ is the velocity of the point due to the rotation $\bar{\omega}$. Ignoring $\bar{v}_{cg}$, we find for the components of $\bar{v}$:

$$\bar{v} = q\delta_z - r\delta_y, \quad \dot{\delta}_y + rx - p\delta_z, \quad \dot{\delta}_z + p\delta_y - qx \tag{2}$$

and for its magnitude

$$\begin{aligned} |\bar{v}|^2 &= (p^2+q^2)\delta_z^2 + (p^2+r^2)\delta_y^2 + \dot{\delta}_y^2 + \dot{\delta}_z^2 + x^2(q^2+r^2) \\ &\quad - 2qr\delta_y\delta_z - 2px(q\delta_y + r\delta_z) + 2p(\delta_y\dot{\delta}_z - \dot{\delta}_y\delta_z) \\ &\quad + 2x(r\dot{\delta}_y - q\dot{\delta}_z) \end{aligned} \tag{3}$$

Lagrange's equation, including damping, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i \tag{4}$$

where q_i are generalized coordinates and Q_i are generalized forces. The kinetic energy T consists of rotational plus translational energy and is written

$$T = \frac{1}{2}I_x p^2 + \frac{1}{2}\int_L |\bar{v}|^2 m(x)\,\mathrm{d}x \tag{5}$$

where the integration extends over the length of the missile and $m(x)$ is the mass per unit length. The other rotational terms $\frac{1}{2}Iq^2 + \frac{1}{2}Ir^2$ are contained in the integral of Eq. (5) through the rigid-body terms $x^2(q^2+r^2)$ in Eq. (3). The potential energy U includes only the elastic strain energy from flexure, which can be written (see, for example, Langhaar[7])

$$U = \frac{1}{2}\int_L EI\left[\left(\frac{\partial^2\delta_y}{\partial x^2}\right)^2 + \left(\frac{\partial^2\delta_z}{\partial x^2}\right)^2\right]\mathrm{d}x \tag{6}$$

Damping energy D is given by

$$D = \frac{1}{2}\int_L c(\dot{\delta}_y^2 + \dot{\delta}_z^2)\,\mathrm{d}x \tag{7}$$

The deflections δ_y, δ_z are expanded in terms of the normal modes $\phi_i(x)$ of the structure according to

$$\begin{aligned} \delta_y(x,t) &= \sum_i \phi_i(x)\eta_i(t) \\ \delta_z(x,t) &= \sum_i \phi_i(x)\zeta_i(t) \end{aligned} \tag{8}$$

and we take for the generalized coordinates η_i and ζ_i. The normal modes must satisfy, from the beam equation, the differential equation[8]

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left[EI\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi_i(x)\right] = \omega_i^2 m(x)\phi_i(x) \tag{9}$$

and the boundary conditions, where ω_i is the ith mode bending frequency. The normal modes also satisfy the orthogonality conditions

$$\int_L \phi_i(x)\phi_j(x)m(x)\,\mathrm{d}x = \begin{cases} 0, & i \neq j \\ M_i, & i = j \end{cases} \tag{10}$$

and the free-free beam condition

$$\int_L x\phi_i(x)m(x)\,\mathrm{d}x = 0 \tag{11}$$

where M_i in Eq. (10) is the generalized mass. To evaluate Lagrange's equation, we need to express the energy equations, Eqs. (5-7), in terms of the generalized coordinates η_i and ζ_i through Eq. (8). Consider, for example, the term $\dot{\delta}_y^2$ in Eq. (3) that must be integrated over the missile length in Eq. (5). We obtain for this term using the orthogonality condition, Eq. (10),

$$\begin{aligned} \int_L \dot{\delta}_y^2(x,t)m(x)\,\mathrm{d}x &= \sum_i\sum_j \dot{\eta}_i\dot{\eta}_j \int_L \phi_i(x)\phi_j(x)m(x)\,\mathrm{d}x \\ &= \sum_i M_i\dot{\eta}_i^2 \end{aligned} \tag{12}$$

Similarly, for the other terms in Eq. (3) we obtain for the integral in the kinetic energy, Eq. (5),

$$\begin{aligned} \int_L |\bar{v}|^2 m(x)\,\mathrm{d}x &= (p^2+q^2)\sum_i M_i\zeta_i^2 + (p^2+r^2)\sum_i M_i\eta_i^2 \\ &\quad + \sum_i M_i(\dot{\eta}_i^2 + \dot{\zeta}_i^2) + 2p\sum_i M_i(\eta_i\dot{\zeta}_i - \dot{\eta}_i\zeta_i) - 2qr\sum_i M_i\eta_i\zeta_i \\ &\quad + (q^2+r^2)\int_L x^2 m(x)\,\mathrm{d}x \end{aligned} \tag{13}$$

The potential energy, Eq. (6), can be written

$$U = \frac{1}{2}\sum_i\sum_j(\eta_i\eta_j + \zeta_i\zeta_j)\int_L EI\phi_i''(x)\phi_j''(x)\,\mathrm{d}x \tag{14}$$

By two successive integrations by parts starting with $u = EI\phi_i''(x)$, $\mathrm{d}v = \phi_j''(x)\,\mathrm{d}x$, it can be shown that Eq. (14) in conjunction with Eqs. (9) and (10) reduces to

$$U = \frac{1}{2}\sum_i(\eta_i^2 + \zeta_i^2)\omega_i^2 M_i \tag{15}$$

The damping energy, Eq. (7), can be written

$$D = \sum_i \mu_i\omega_i M_i(\dot{\eta}_i^2 + \dot{\zeta}_i^2) \tag{16}$$

where we have replaced c by $2\mu_i\omega_i$, μ_i being the critical damping ratio of the ith mode. To evaluate Lagrange's equation, we also need the generalized force Q_i, which is determined from the work done by the applied aerodynamic forces in a virtual displacement q_i. If $f_y(x,t)$ is the distributed load acting in the y direction, then the work done in the virtual displacement $\delta\eta_i$, from Eq. (8), is

$$\delta W_y = \int_L f_y(x,t) \sum_i \phi_i(x)\delta\eta_i(t)\,\mathrm{d}x$$

$$= \sum_i \delta\eta_i(t) \int_L f_y(x,t)\phi_i(x)\,\mathrm{d}x \tag{17}$$

in which the generalized force is

$$Q_i^y = \int_L f_y(x,t)\phi_i(x)\,\mathrm{d}x \tag{18}$$

Similarly, for the virtual displacement ζ_i,

$$Q_i^z = \int_L f_z(x,t)\phi_i(x)\,\mathrm{d}x \tag{19}$$

If we perform the indicated differentiations in Lagrange's equation, Eq. (4), with $q_i = \eta_i$, we obtain

$$\ddot{\eta}_i + 2\mu_i\omega_i\dot{\eta}_i - 2p\dot{\zeta}_i + (\omega_i^2 - p^2 - r^2)\eta_i + qr\zeta_i$$

$$= \frac{1}{M_i}\int_L f_y(x,t)\phi_i(x)\,\mathrm{d}x \tag{20}$$

Similarly, with $q_i = \zeta_i$,

$$\ddot{\zeta}_i + 2\mu_i\omega_i\dot{\zeta}_i + 2p\dot{\eta}_i + (\omega_i^2 - p^2 - q^2)\zeta_i + qr\eta_i$$

$$= \frac{1}{M_i}\int_L f_z(x,t)\phi_i(x)\,\mathrm{d}x \tag{21}$$

Defining the complex parameters

$$\delta_i(t) = \eta_i(t) + i\zeta_i(t), \qquad f(x,t) = f_y(x,t) + if_z(x,t) \tag{22}$$

we obtain for the ith mode deflection equation

$$\ddot{\delta}_i + (2\mu_i\omega_i + 2ip)\dot{\delta}_i + (\omega_i^2 - p^2 + qr)\delta_i - r^2\eta_i - iq^2\zeta_i$$

$$= \frac{1}{M_i}\int_L f(x,t)\phi_i(x)\,\mathrm{d}x \tag{23}$$

The actual deflections, from the summation of modes, Eq. (8), are

$$\delta(x,t) = \delta_y(x,t) + i\delta_z(x,t) = \sum_i \delta_i(t)\phi_i(x) \tag{24}$$

B. Angular Rates

The angular rate equations are derived after Meirovitch and Nelson[1] in terms of quasicoordinates,[9] which are angular coordinates representing rotation components about the orthogonal body-fixed axes x,y,z. The corresponding equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial q}\right) - p\frac{\partial T}{\partial r} + r\frac{\partial T}{\partial p} = M_y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial r}\right) - q\frac{\partial T}{\partial r} + p\frac{\partial T}{\partial p} = M_z$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial p}\right) - r\frac{\partial T}{\partial q} + q\frac{\partial T}{\partial r} = M_x \tag{25}$$

where M_x, M_y, M_z are moments about x, y, z respectively. Substitution of the kinetic energy, Eq. (5), into the first two lines of Eq. (25) yields for the lateral moments

$$M_y = \dot{q}\left(I + \sum_i M_i\zeta_i^2\right) + pr\left(-I + I_x + \sum_i M_i\zeta_i^2\right)$$

$$+ (pq - \dot{r})\sum_i M_i\eta_i\zeta_i + 2q\sum_i M_i\zeta_i\dot{\zeta}_i - 2r\sum_i M_i\zeta_i\dot{\eta}_i \tag{26}$$

$$M_z = \dot{r}\left(I + \sum_i M_i\eta_i^2\right) + pq\left(I - I_x - \sum_i M_i\eta_i^2\right)$$

$$- (pr + \dot{q})\sum_i M_i\eta_i\zeta_i + 2r\sum_i M_i\eta_i\dot{\eta}_i - 2q\sum_i M_i\eta_i\dot{\zeta}_i \tag{27}$$

where, by definition, the lateral moment of inertia I is the integral

$$I = \int_L x^2 m(x)\,\mathrm{d}x \tag{28}$$

The bending deflection terms in Eqs. (26) and (27) give the appearance of moments or products of inertia, consisting of products of generalized mass and deflection squared. The first two such terms in Eqs. (26) and (27) add to or subtract from the lateral moment of inertia. If we define the complex lateral rate

$$\Omega = q + ir \tag{29}$$

then the moment equations, including all nonlinear terms, can be written

$$\frac{M_y + iM_z}{I} = \dot{\Omega} - \frac{i}{I}[\dot{q} - i\dot{r} - ip(q - ir)]\sum_i M_i\eta_i\zeta_i$$

$$+ \frac{1}{I}\sum_i M_i[(\dot{q} + pr)\zeta_i^2 + i(\dot{r} - pq)\eta_i^2]$$

$$+ i\left(1 - \frac{I_x}{I}\right)p\Omega - \frac{2i}{I}\sum_i M_i\delta_i(q\dot{\zeta}_i - r\dot{\eta}_i) \tag{30}$$

C. Angle of Attack

By analogy with the equations of rotational motion, Eq. (25), we can write similar expressions in terms of kinetic energy for the translational equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial v}\right) - p\left(\frac{\partial T}{\partial w}\right) + r\left(\frac{\partial T}{\partial u}\right) = F_y$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial w}\right) - q\left(\frac{\partial T}{\partial u}\right) + p\left(\frac{\partial T}{\partial v}\right) = F_z \tag{31}$$

where F_y, F_z are lateral forces in the y,z directions, and u,v,w are components of the velocity $\bar{v}_{cg}$ of the moving origin in Eq. (1). If we add u,v,w to the velocity components in Eq. (2) and substitute in Eq. (5), the additional terms in the kinetic energy from cg translation are

$$\Delta T = \frac{1}{2}m(u^2 + v^2 + w^2) + u\int_L (q\delta_z - r\delta_y)m(x)\,\mathrm{d}x$$

$$+ v\int_L (\dot{\delta}_z + rx - p\delta_z)m(x)\,\mathrm{d}x$$

$$+ w\int_L (\dot{\delta} + p\delta_y - qx)m(x)\,\mathrm{d}x \tag{32}$$

It can be shown from conditions of orthogonality that these terms do not contribute to either the bending or angular rate equations derived earlier. Performing the derivatives indicated in Eq. (31), we can write the translational equations of motion in terms of the complex angle of attack ξ, defined by

$$\xi = \beta + i\alpha \tag{33}$$

where angle of attack α and angle of sideslip β are, for small angles,

$$\alpha = w/u, \qquad \beta = v/u \tag{34}$$

The equations in complex notation are

$$\dot{\xi} + ip\xi - i\Omega - \frac{1}{mu}\int_L (\ddot{\delta} + 2ip\dot{\delta} - p^2\delta)m(x)\,\mathrm{d}x$$

$$-\frac{i\Omega}{mu}\int_L (q\delta_z - r\delta_y)m(x)\,\mathrm{d}x = \frac{F_y + iF_z}{mu} \tag{35}$$

where $m = \int_L m(x)\,\mathrm{d}x$ is the vehicle mass. The integral terms containing δ in Eq. (35) are identically zero from orthogonality with the rigid-body modes. The translational motion or angle-of-attack equations then reduce to

$$\dot{\xi} + ip\xi - i\Omega = \frac{F_y + iF_z}{mu} \tag{36}$$

The coupled system of equations, Eqs. (23), (30), and (36), describes the missile aeroelastic behavior. The aerodynamic forces and moments, which are functions of the state variables, are derived in the following section.

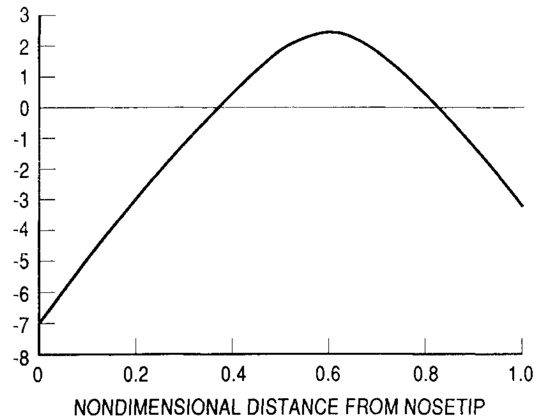

Fig. 3 First free-free bending mode shape.

D. Aerodynamic Model

1. Moment Equations

The moments on the right-hand side of Eq. (30) are written as the summation of moments due to lift on each incremental section of the missile. For the component M_y,

$$M_y = \int_L xL_\alpha(x)\alpha(x,t)\,\mathrm{d}x + M_y^T \tag{37}$$

where M_y^T is a resultant body-fixed trim moment, $L_\alpha(x)$ is the lift force derivative per unit length, and $\alpha(x,t)$ is the effective angle of attack along the missile. The angle of attack is composed of the rigid-body value α, the bending slope $\delta_z'(x,t)$, and the damping contributions from deflection rate $\dot{\delta}_z(x,t)$ and angular rate q according to

$$\alpha(x,t) = \alpha - \delta_z'(x,t) + \dot{\delta}_z(x,t)/u - qx/u$$

$$= \alpha - \sum_i \phi_i'(x)\zeta_i(t) + \frac{1}{u}\sum_i \phi_i(x)\dot{\zeta}_i(t) - qx/u \tag{38}$$

Substituting Eq. (38) into Eq. (37), we obtain for M_y

$$M_y = \alpha\int_L xL_\alpha(x)\,\mathrm{d}x - \sum_i \zeta_i(t)\int_L xL_\alpha(x)\phi_i'(x)\,\mathrm{d}x$$

$$+\frac{1}{u}\sum_i \dot{\zeta}_i(t)\int_L xL_\alpha(x)\phi_i(x)\,\mathrm{d}x$$

$$-\frac{q}{u}\int_L x^2L_\alpha(x)\,\mathrm{d}x + M_y^T \tag{39}$$

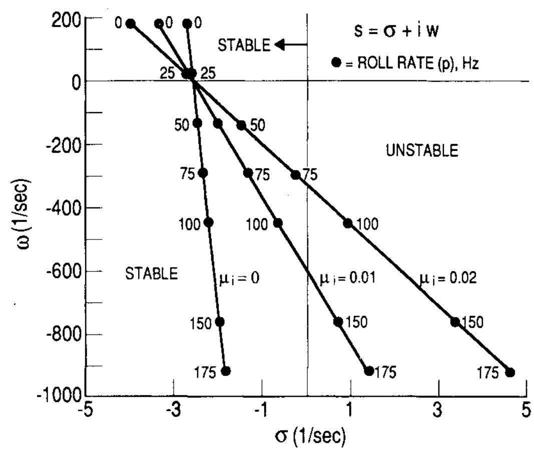

Fig. 2 Dominant root locus vs roll rate (p) and structural damping ratio (μ_i), with three bending modes.

We can write a similar expression for M_z in terms of $L_\beta(x)$ that, for the axially symmetric missile $L_\beta = L_\alpha$, combines with Eq. (39) to give the complex moment

$$\frac{M_y + iM_z}{I} = -C^*_{m_q}\Omega + i\omega_\xi^2 - \sum_i a_3^i\dot{\delta}_i - \sum_i a_4^i\delta_i + \frac{M_T}{I} \tag{40}$$

where

$$\omega^2 \equiv -\frac{1}{I}\int_L xL_\alpha(x)\,\mathrm{d}x \tag{41}$$

$$C^*_{m_q} \equiv \frac{1}{Iu}\int_L x^2L_\alpha(x)\,\mathrm{d}x \equiv -C_{m_q}q_\infty Sd^2/2Iu \tag{42}$$

$$a_3^i \equiv \frac{i}{Iu}\int_L xL_\alpha(x)\phi_i(x)\,\mathrm{d}x \tag{43}$$

$$a_4^i \equiv -\frac{i}{I}\int_L xL_\alpha(x)\phi_i'(x)\,\mathrm{d}x \tag{44}$$

and M_T is the complex trim moment $M_y^T + iM_z^T$. The integral term in Eq. (42) is based on the assumption that C_{m_q} can be obtained from static aerodynamics. For a better approximation of C_{m_q} either from dynamic stability measurements or from flight data, the second expression in Eq. (42) should be used.

2. Lateral Forces

The lateral forces on the right-hand side of Eq. (36) are the resultants of lift contributions along the missile in the y and z directions. The axial distribution of local angle of attack, including bending deformation and damping, is given in Eq. (38). On multiplying this by the lift force derivative per unit length and integrating over the missile, we obtain for the lateral force F_z

$$F_z = -\alpha\int_L L_\alpha(x)\,\mathrm{d}x + \sum_i \zeta_i(t)\int_L L_\alpha(x)\phi_i'(x)\,\mathrm{d}x$$

$$-\frac{1}{u}\sum_i \dot{\zeta}_i(t)\int_L L_\alpha(x)\phi_i(x)\,\mathrm{d}x + \frac{q}{u}\int_L xL_\alpha(x)\,\mathrm{d}x + F_z^T \tag{45}$$

where we have added the resultant body-fixed trim force in the z direction F_z^T. We obtain a similar expression for F_y, and the complex sum divided by mu is given by

$$\frac{F_y + iF_z}{mu} = -C^*_{N_\alpha}\xi - ib\Omega - \sum_i (b_3^i\dot{\delta}_i - b_4^i\delta_i) + \frac{F_T}{mu} \tag{46}$$

where $F_T \equiv F_y^T + iF_z^T$ and the coefficients are defined by

$$C_{N_\alpha}^* \equiv \frac{1}{mu}\int_L L_\alpha(x)\,\mathrm{d}x \tag{47}$$

$$b \equiv -\frac{1}{mu^2}\int_L xL_\alpha(x)\,\mathrm{d}x \tag{48}$$

$$b_3^i \equiv \frac{1}{mu^2}\int_L L_\alpha(x)\phi_i(x)\,\mathrm{d}x \tag{49}$$

$$b_4^i \equiv -\frac{1}{mu}\int_L L_\alpha(x)\phi_i'(x)\,\mathrm{d}x \tag{50}$$

3. *Aeroelastic Deflections*

The aerodynamic forcing term on the right-hand side of the deflection equation, Eq. (23), contains the product of the incremental force per unit length $f(x,t)$ and the ith mode shape $\phi_i(x)$. The component $f_z(x,t)$ is

$$f_z(x,t) = -L_\alpha(x)\alpha(x,t) \tag{51}$$

where $\alpha(x,t)$ is defined by Eq. (38). If we multiply Eq. (51) by $\phi_i(x)$ and integrate over the length of the missile, we obtain

$$\begin{aligned}\int_L f_z(x,t)\phi_i(x)\,\mathrm{d}x &= -\alpha\int_{L_\alpha}(x)\phi_i(x)\,\mathrm{d}x \\ &+ \sum_j \zeta_j(t)\int_L L_\alpha(x)\phi_i(x)\phi_j'(x)\,\mathrm{d}x \\ &- \frac{1}{u}\sum_j \dot{\zeta}_j(t)\int_L L_\alpha(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x + \frac{q}{u}\int_L xL_\alpha(x)\phi_i(x)\,\mathrm{d}x \\ &+ \int_L f_z^T(x,t)\phi_i(x)\,\mathrm{d}x \end{aligned} \tag{52}$$

where $f_z^T(x,t)$ is a body-fixed trim force. Writing a similiar expression for $\int_L f_y(x,t)\phi_i(x)\,\mathrm{d}x$ and again taking the complex sum with the condition of axial symmetry $L_\beta = L_\alpha$, we obtain

$$\begin{aligned}\int_L f(x,t)\phi_i(x)\,\mathrm{d}x &= -c_3^i\Omega - c_4^i\xi - \sum_j e_1^{ij}\dot{\delta}_j(t) \\ &- \sum_j e_2^{ij}\delta_j(t) + \frac{1}{M_i}\int_L f_T(x,t)\phi_i(x)\,\mathrm{d}x \end{aligned} \tag{53}$$

where

$$c_3^i \equiv -\frac{i}{uM_i}\int_L xL_\alpha(x)\phi_i(x)\,\mathrm{d}x \tag{54}$$

$$c_4^i \equiv \frac{1}{M_i}\int_L L_\alpha(x)\phi_i(x)\,\mathrm{d}x \tag{55}$$

$$e_1^{ij} \equiv \frac{1}{uM_i}\int_L L_\alpha(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x \tag{56}$$

$$e_2^{ij} \equiv -\frac{1}{M_i}\int_L L_\alpha(x)\phi_i(x)\phi_j'(x)\,\mathrm{d}x \tag{57}$$

$$f_T(x,t) = f_y^T(x,t) + if_z^T(x,t) \tag{58}$$

4. *Equations of Motion—Summary*

Equations (30), (36), and (23) with the corresponding aerodynamic forcing functions Eqs. (40), (46), and (53), respectively, describe the missile aeroelastic behavior. The complete coupled equations including all nonlinear terms are

$$\begin{aligned}\dot{\Omega} &- \frac{i}{I}[\dot{q} - i\dot{r} + ip(q - ir)]\sum_i M_i\eta_i\zeta_i + \frac{1}{I}\sum_i M_i[(\dot{q} + pr)\zeta_i^2 \\ &+ i(\dot{r} - pq)\eta_i^2] + \left[i\left(1 - \frac{I_x}{I}\right)p + c_{m_q}^*\right]\Omega - i\omega^2\xi \\ &+ \sum_i(a_3^i\dot{\delta}_i + a_4^i\delta_i) - \frac{2i}{I}\sum_i M_i\delta_i(q\dot{\zeta}_i - r\dot{\eta}_i) = \frac{M_T}{I}\end{aligned} \tag{59}$$

$$\dot{\xi} + (ip + C_{N_\alpha}^*)\xi - i(1 - b)\Omega + \sum_i(b_3^i\dot{\delta}_i + b_4^i\delta_i) = \frac{F_T}{mu} \tag{60}$$

$$\begin{aligned}&\ddot{\delta}_i + (2\mu_i\omega_i + 2ip)\dot{\delta}_i + \sum_j(e_1^{ij}\dot{\delta}_j + e_2^{ij}\delta_j) \\ &\quad + (\omega_i^2 + qr - p^2)\delta_i - r^2\eta_i - iq^2\zeta_i + c_3^i\Omega + c_4^i\xi \\ &\quad = \frac{1}{M_i}\int_L f_T(x,t)\phi_i(x)\,\mathrm{d}x \\ &\qquad\vdots \qquad\qquad\qquad\qquad \vdots\end{aligned} \tag{61}$$

No attempt is made here to assess the relative importance of the nonlinear terms. That could constitute a topic for a future publication. If we omit the nonlinear terms, the coupled equations can be written as the linear set using the more abbreviated nomenclature

$$\dot{\Omega} + a_1\Omega + a_2\xi + \sum_i(a_3^i\dot{\delta}_i + a_4^i\delta_i) = \frac{M_T}{I} \tag{62}$$

$$b_2\Omega + \dot{\xi} + b_1\xi + \sum_i(b_3^i\dot{\delta}_i + b_4^i\delta_i) = \frac{F_T}{mu} \tag{63}$$

$$\begin{aligned}&c_3^i\Omega + c_4^i\xi + \ddot{\delta}_i + c_1^i\dot{\delta}_i + c_2^i\delta_i + \sum_j(e_1^{ij}\dot{\delta}_j + e_2^{ij}\delta_j) \\ &\qquad = \frac{1}{M}\int_L f_T(x,t)\phi_i(x)\,\mathrm{d}x \\ &\qquad\vdots \qquad\qquad\qquad\qquad \vdots\end{aligned} \tag{64}$$

where

$$a_1 \equiv i\left(1 - \frac{I_x}{I}\right)p + C_{m_q}^* \qquad b_1 \equiv ip + C_{N_\alpha}^*$$

$$a_2 \equiv -i\omega^2 \qquad b_2 \equiv -i(1 - b)$$

$$a_3^i \equiv -iI_2^i/Iu \qquad b_3^i \equiv I_1^i/mu^2$$

$$a_4^i \equiv -iI_4^i/I \qquad b_4^i \equiv -I_3^i/mu$$

$$b \equiv M_\alpha/mu^2 \qquad c_1^i \equiv 2\mu_i\omega_i + 2ip$$

$$c_2^i \equiv \omega_i^2 - p^2 \qquad M_\alpha \equiv \int_L xL_\alpha(x)\,\mathrm{d}x$$

$$c_3^i \equiv -iI_2^i/uM_i \qquad M_i \equiv \int_L \phi_i^2(x)m(x)\,\mathrm{d}x$$

$$c_4^i \equiv I_1^i/M_i \qquad I_1^i \equiv \int_L L_\alpha(x)\phi_i(x)\,\mathrm{d}x$$

$$e_1^{ij} \equiv I_5^{ij}/uM_i \qquad I_2^i \equiv \int_L xL_\alpha(x)\phi_i(x)\,\mathrm{d}x$$

$$e_2^{ij} \equiv -I_6^{ij}/M_i \qquad I_3^i \equiv \int_L L_\alpha(x)\phi_i'(x)\,\mathrm{d}x$$

$$C_{N_\alpha}^* \equiv L_\alpha/mu \qquad I_4^i \equiv \int_L xL_\alpha(x)\phi_i'(x)\,\mathrm{d}x$$

$$\omega^2 \equiv M_\alpha/I \qquad I_5^{ij} \equiv \int_L L_\alpha(x)\phi_i(x)\phi_j(x)\,\mathrm{d}x$$

$$L_\alpha \equiv \int_L L_\alpha(x)\,\mathrm{d}x \qquad I_6^{ij} \equiv \int_L L_\alpha(x)\phi_i(x)\phi_j'(x)\,\mathrm{d}x$$

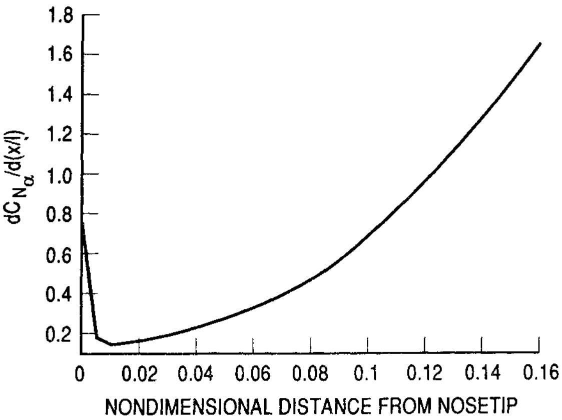

Fig. 4 Nosetip lift-force derivative.

There is a bending deflection equation for each ith bending mode, with $i = 1$ the first (lowest) bending mode. The Laplace transform of Eqs. (62–64) in terms of the complex variable s is written

$$[A]\begin{Bmatrix} \Omega \\ \xi \\ \delta_1 \\ \delta_2 \\ \vdots \end{Bmatrix} = \{F\} \tag{65}$$

$$[A] = \begin{bmatrix} s+a_1 & a_2 & a_3^1 s + a_4^1 & a_3^2 s + a_4^2 & \cdots \\ b_2 & s+b_1 & b_3^1 s + b_4^1 & b_3^2 s + b_4^2 & \cdots \\ c_3^1 & c_4^1 & s^2 + (c_1^1 + e_1^{11})s + c_2^1 + e_2^{11} & e_1^{12}s + e_2^{12} & \cdots \\ c_3^2 & c_4^2 & e_1^{21}s + e_2^{21} & s^2 + (c_1^2 + e_1^{22})s + c_2^2 + e_2^{22} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{66}$$

$$\{F\} = \begin{Bmatrix} \dfrac{M_T(s)}{I} \\ \dfrac{F_T(s)}{mu} \\ \dfrac{1}{M_1}\displaystyle\int_L f_T(x,s)\phi_1(x)\,\mathrm{d}x \\ \dfrac{1}{M_2}\displaystyle\int_L f_T(x,s)\phi_2(x)\,\mathrm{d}x \\ \vdots \end{Bmatrix} \tag{67}$$

E. Aeroelastic Stability

Stability is determined from the matrix $[A]$, Eq. (66), the determinant of which set equal to zero yields the system characteristic equation in the complex variable $s = \sigma + i\omega$. With only a single elastic bending mode, the characteristic equation is a fourth-order polynomial in s. Each additional bending mode increases by two the order of the complex characteristic equation. For example, with three bending modes, the characteristic equation is an eighth-order polynomial in s. Dynamic stability is determined from the complex roots of the characteristic equation. Stability requires that all roots have negative real parts. Static stability is readily determined by setting $s = 0$. Particular cases of interest are values of roll rate and dynamic pressure that produce static aeroelastic divergence. Consider the single-mode solution $i = 1$ and neglect for simplicity all damping terms, i.e., containing the velocity u in the denominator. The matrix $[A]$ with $s = 0$ reduces to

$$[A] = \begin{bmatrix} a_1 & a_2 & a_4 \\ b_2 & b_1 & 0 \\ 0 & c_4 & c_2 + e_2 \end{bmatrix} \tag{68}$$

where we have dropped the superscripts for the single-mode solution. The determinant of $[A]$ set equal to zero yields

$$\left[\omega^2 - \left(1 - \frac{I_x}{I}\right)p^2\right]\left[\omega_1^2 - p^2 - \frac{I_6}{M_1}\right] - \frac{I_1 I_4}{I M_1} = 0 \tag{69}$$

a solution of which gives critical values of the roll rate p that cause static divergence. The first expression in brackets, if set equal to zero, is recognized to be the rigid-body critical roll rate for pitch-roll coupling or roll resonance, $p_{\rm crit} = \pm\omega/[1 - (I_x/I)]^{1/2}$, where ω is the rigid-body aerodynamic pitch frequency of the nonrolling missile (see, for example, Platus[10]). One root of the quadratic equation for p^2 in Eq. (69) is this critical roll rate as modified by aeroelasticity. The second root is a resonance associated with the first bending frequency of the missile. Because bending reduces the pitch moment at angle of attack and therefore reduces the pitch frequency relative to that for a rigid missile, the critical roll rate for pitch-roll coupling is less than the rigid-body value for the aeroelastic missile. Therefore, if the first bending frequency ω_1 in Eq. (69) is somewhat greater than the rigid-body pitch frequency, then the root $p = p_c$ corresponding to pitch-roll coupling can be ignored in the second bracket in Eq. (69) ($p_c^2 \ll \omega_i^2$) and a good approximation for p_c is

$$p_c = \left\{p_{\rm crit}^2 - \frac{(I_1 I_4/I M_1)}{[1 - (I_x/I)][\omega_1^2 - (I_6/M_1)]}\right\}^{1/2} \tag{70}$$

We can also obtain from Eq. (69) the limiting values of dynamic pressure that produce aeroelastic divergence. The integrals I_1, I_4, I_6 and the frequency term ω^2 are all proportional to the lift force derivative L_α that is proportional to the dynamic pressure q_∞. If we factor out q_∞ and redefine these parameters

$$I_i \equiv q_\infty \mathcal{I}_i$$

$$\omega^2 \equiv q_\infty \omega_0^2 \tag{71}$$

We obtain a quadratic equation in q_∞ for a specified roll rate. For $p = 0$, the limiting dynamic pressure $q_\infty^{\rm div}$ is

$$q_\infty^{\rm div} = \frac{M_1 \omega_1^2}{(\mathcal{I}_1 \mathcal{I}_4 / I \omega_0^2) + \mathcal{I}_6} \tag{72}$$

III. Numerical Examples

Application of the results presented here to assess the aeroelastic stability of a flexible missile under specified flight conditions is a tedious procedure and requires auxiliary computer programming for all but the most idealized cases. Free-free

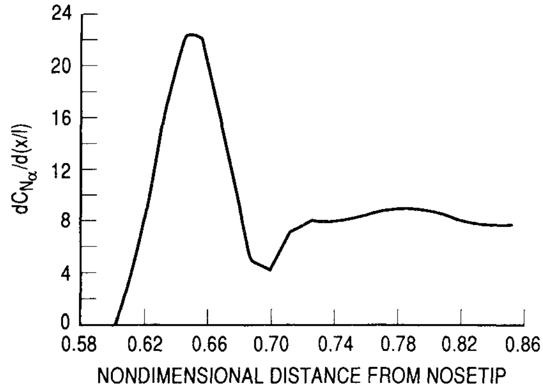

Fig. 5 Flare lift-force derivative.

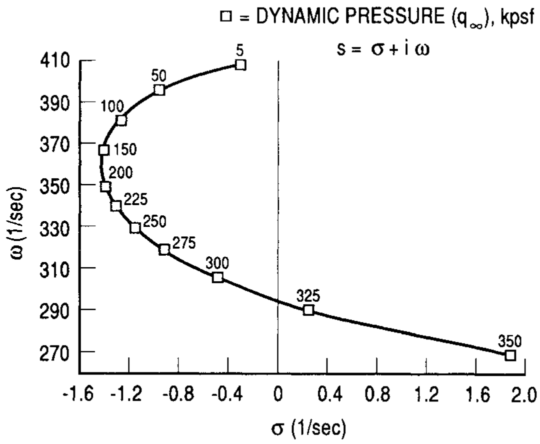

Fig. 6 Dominant root locus vs dynamic pressure.

bending mode shapes and frequencies must be determined for axial distributions (usually nonuniform) of mass and stiffness. The axial distribution of the incremental lift-force derivative dC_{N_α}/dx is also required. The integrals I_k^i and I_k^{ij} of various products of the mode shape, mode-shape derivatives, and incremental lift-force derivatives must be evaluated for the number of modes selected. Finally, a root locus analysis of the complex roots of the characteristic equation must be performed to assess stability.

The first example is for a slender hypersonic missile assumed to have flexural properties that can be approximated by those for a uniform free-free beam. The in vacuo bending frequencies are 135, 372, and 730 Hz, respectively, for the first three bending modes, and the rigid-body aerodynamic pitch frequency (approximately equal to the rigid-body critical roll rate for $I_x \ll I$) is 57 Hz. The effect of static aeroelasticity, characterized by Eqs. (69) and (70) for a single bending mode, reduces the critical roll rate for pitch-roll coupling to 25 Hz. Dynamic stability results are shown in Fig. 2 and consist of root locus plots vs roll rate and viscous structural damping for the dominant root of the characteristic equation obtained from the $[A]$ matrix, Eq. (66), with three bending modes. The results indicate a significant destabilizing effect of structural damping for supercritical roll rates (i.e., for roll rates greater than the critical roll rate for pitch-roll coupling) and a stabilizing effect for subcritical roll rates. With 1% of critical damping, the missile is dynamically unstable for roll rates in excess of approximately 125 Hz, whereas with 2% of critical damping, the missile becomes unstable for roll rates in excess of approximately 80 Hz. For subcritical roll rates ($p < 25$ Hz), the curves cross over so that structural damping has a stabilizing effect; i.e., the real parts of the roots become more negative with increasing damping. The influence of the number of bending modes was examined by evaluating the root locus with the first two bending modes and with only the lowest bending mode. The differences in the root loci were insignificant over the range of roll rate and structural damping shown in Fig. 2 and substantiate the result of Oberholtzer et al.[2] that only the first bending mode has a significant influence on dynamic stability for the configurations analyzed.

A second example illustrates the numerical analysis generally required when the missile mass, stiffness, and aerodynamic properties are nonuniform along the missile. The missile configuration consists of a conical nose, a cylindrical region, and an aft flare, and the axial distribution of mass and stiffness yields the first free-free bending mode shape shown in Fig. 3. Only the first bending mode is used in this example, which yields a fourth-order characteristic equation in the complex variable s. Axial distributions of lift-force derivative along the nosetip and flare regions are shown in Figs. 4 and 5, respectively. The lift is assumed to be negligible along the cylinder. The coefficients contained in the $[A]$ matrix, Eq. (66), are evaluated in terms of the various integrals summarized following Eq. (64). The integrals must be evaluated numerically and consist of various products of the lift-force derivative with the mode shape, its derivative, and the axial distance to the center of gravity. The determinant of $[A]$ yields the characteristic equation in the complex variable s, and a root locus of the dominant root is shown in Fig. 6 vs dynamic pressure, which describes the system dynamic aeroelastic stability. Root loci are also shown vs roll rate in Fig. 7 for 0, 1, and 2% of critical structural damping, respectively. The re-

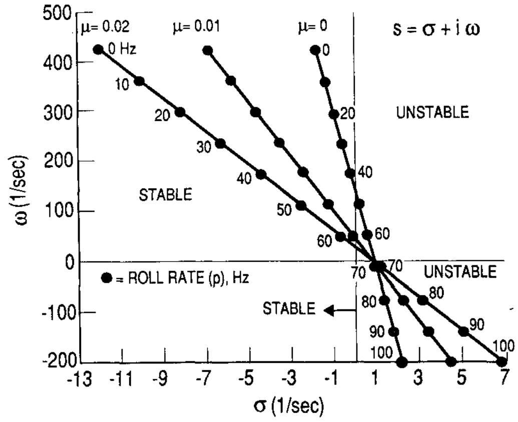

Fig. 7 Dominant root locus vs roll rate (p) and structural damping ratio (μ), with a single bending mode.

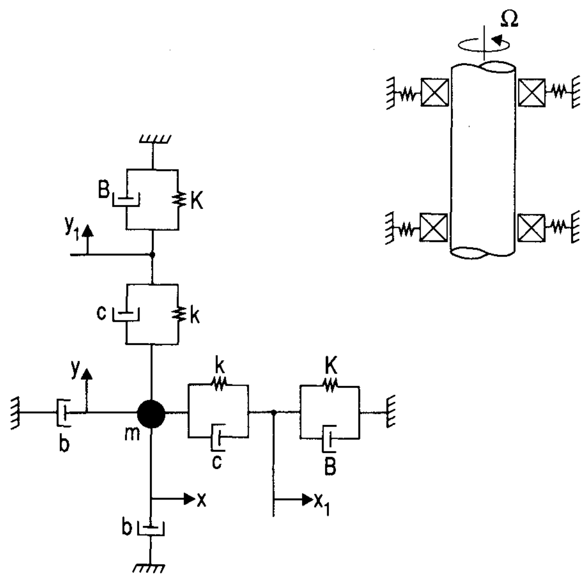

Fig. 8 Lumped-mass rotor model (bending displacements x,y; support displacements x_1, y_1; viscous internal damping c; external damping b,B).

sults show a similar destabilizing influence of structural damping as in the first example. Damping is stabilizing at subcritical roll rates (in this case below approximately 70 Hz) and destabilizing at supercritical roll rates.

IV. Interpretation of Results

It is well known that the presence of damping can destabilize dynamic systems. This occurs when the phasing is such that the dynamic damping forces or moments act in a direction such to excite or reinforce the motion rather than to impede it. A particularly relevant example and one that appears to explain the results observed here is the destabilizing influence of damping on whirl phenomena in rotating machinery. It is shown by Lund[11] that viscous structural damping of the alternating flexure of a rotating shaft suspended in compliant bearings will induce a whirl instability at certain rotation speeds. The physical system and the conditions required for the instability are strikingly similar to the rotating aeroelastic missile and the dynamic stability behavior illustrated in Figs. 2 and 7. The rotor model is described schematically in Fig. 8 and consists of a symmetric shaft with a bending stiffness k and a lumped mass m. The shaft is supported at the ends in compliant bearings of stiffness K such that the natural frequency of the system is

$$\omega_0 = \sqrt{\frac{kK}{m(k+K)}} \tag{73}$$

If the flexural motion of the shaft has an internal viscous damping constant c and external damping constants b and B for shaft flexure and support motion, respectively, Lund[11] has obtained the condition for stability, assuming c, b, and B are all small quantities,

$$\left(\frac{K+k}{k}\right)^2 \omega_0 b + \left(\frac{k}{K}\right)^2 \omega_0 B > (\Omega - \omega_0)c \tag{74}$$

where Ω is the shaft rotation rate. With viscous damping c, instability can occur only for shaft rotation rates above the critical frequency ω_0. Stable operation above the critical frequency requires the external damping given by Eq. (74).

The analogy with the spinning aeroelastic missile is readily apparent. The natural frequency ω_0 of the rotor system corresponds with the natural pitch frequency or critical roll rate of the spinning missile. Compliance in the rotor support bearings acts in series with shaft flexure to reduce the whirl frequency according to Eq. (73), whereas missile flexibility acts to reduce the aerodynamic pitch moment and therefore reduces the rigid-body pitch frequency or critical roll rate according to Eqs. (69) or (70). External damping of the rotor shaft and bearings corresponds with aerodynamic pitch and normal force damping $C^*_{m_q}$ and $C^*_{N_\alpha}$ of the aeroelastic missile. The results illustrated in Figs. 2 and 7 were obtained with both aerodynamic damping and viscous structural damping. Other cases evaluated by holding the structural damping fixed and varying the aerodynamic damping indicate that aerodynamic damping is stabilizing; i.e., an increase in aerodynamic damping shifts the family of root loci in Figs. 2 and 7 to the left. This is quite analogous to the rotor stability condition, Eq. (74), in which external damping is stabilizing. Although the missile aeroelastic model is considerably more complex than the simple lumped-mass rotor model, the rotor model appears to exhibit all of the salient features of the missile dynamic aeroelasticity: 1) viscous internal structural damping is destabilizing only for rotation rates above a critical rotation speed; 2) external rotor and support damping, analogous to missile aerodynamic damping, is stabilizing; and 3) support bearing compliance reduces the fundamental whirl frequency just as missile flexibility reduces the critical roll rate, and it is this reduced critical frequency above which structural damping is destabilizing.

V. Summary and Conclusions

A rigorous derivation of the nonlinear equations of motion that describe the aeroelastic stability of spinning, flexible missiles has been presented. A Lagrangian approach is used that couples the elastic deflections of the missile with parameters that are normally used to describe the rigid-body flight behavior of spinning missiles. The complete nonlinear equations are presented, although no attempt is made to assess the relative importance of the nonlinear terms. This could constitute a topic for future publication. The equations are reduced to a linear set with first-order aerodynamics, which is amenable to a root locus evaluation of static and dynamic stability.

The influence of missile flexibility on static stability was shown to reduce the critical frequency for pitch-roll coupling. The extent of this effect is derived quantitatively for a single elastic bending mode. Also discussed is the limiting dynamic pressure for static aeroelastic divergence. Two examples are included that illustrate the dynamic aeroelastic stability of slender, spinning, hypersonic missiles. Root loci were evaluated as a function of roll rate and viscous structural damping to determine dynamic stability boundaries. It was found for the missile configurations analyzed that the inclusion of two and three bending modes had little influence on stability calculated using only the first bending mode. It was found also that viscous structural damping has a destabilizing effect on stability at roll rates above the critical frequency for roll-pitch coupling. This behavior is explained by analogy with a simple lumped-mass model for the whirl instability of a flexible rotor induced by viscous internal damping. Although not as complex as the spinning, aeroelastic missile, the flexible rotor supported in compliant bearings bears striking similarly to the aeroelastic missile. Results described in the literature based on a simple rotor model exhibit the salient stability features of the spinning aeroelastic missile.

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